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NUMERICAL ANALYSIS

Finite Differences:

Let $y = f(x)$ be a function of x. when the variable x changes in arithmetical progression, say $x = x_0, x_1, x_2, \ldots$ where $x_0 = a + h$, $x_1 = a + 2h, \ldots$, the function f takes the values $y_0 = f(x_0)$, $y_1 = f(x_1)$, $y = f(x_2)$, … . Here h is known as interval of differencing. The value of the independent variable x is known as argument and that of the dependent variable say $y = f(x)$ as entry.

Interpolation & Extrapolation:

The method of estimating the value of a function for any intermediate value of the argument from the given set of values of the function for certain values of the argument is known as interpolation.

On the other hand, the method of estimating the value of the outside the given range is called extrapolation.

Difference Operators:

1. Forward difference operator:

The operator Δ defined by $\Delta f(x) = f(x+h) - f(x)$ called forward difference operator or descending difference operator.

2. Backward Operator:

The operator ∇ defined by $\nabla \mathrm{f(x)} = \mathrm{f(x)} - \mathrm{f(x-h)}$ is called backward difference operator or ascending difference operator.

3. Central difference operator:

The operator δ of f(x) defined by $\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$ called central difference operator.

4. Averaging Operator:

The operator $\mu f(x) = \frac{1}{2}\left\{f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right\}$ called the averaging operator or the mean value operator.

5. Shifting Operator:

The operator $E\, f(x) = f(x+h)$, where h is the interval of differencing, is called the shifting operator or shift operator or displacement operator or translation.

In general, $E^n f(x) = f(x+nh)$ where n is any real number.

Algebraic Properties of operators:

- If f(x) = a, a constant then $\Delta f(x) = 0$ and E f(x) = a.
- If f(x), g(x) are two functions then i) $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$
  ii) $E[f(x) + g(x)] = E f(x) + E g(x)$.
- If f(x) is a function and a is a constant then i) $\Delta[a f(x)] = a \Delta f(x)$ ii) $E[a f(x)] = a E f(x)$.
- If f(x), g(x) are two functions and a, b are two constants then
  i) $\Delta[a f(x) + b g(x)] = a \Delta f(x) + b \Delta g(x)$
  ii) $E[a f(x) + b g(x)] = a E f(x) + b E g(x)$.
- The operators Δ and E are commutative. i.e. $\Delta E = E\Delta$.
- If f(x) , g(x) are two functions then i) $\Delta[f(x)g(x)] = [E f(x)]\Delta g(x) + g(x)\Delta f(x)$
  ii) $\Delta[\,f(x)g(x)\,] = \{\,g(x)\Delta f(x) - f(x)\Delta g(x)\,\} / g(x)Eg(x)$
- If m, n are positive integers then $E^m E^n = E^{m+n}$.

Symbolic Relations and Separation of Symbols:

- $\Delta = E - 1$
- $\nabla = 1 - E^{-1}$
- $E = e^{hD}$
- $(1+\Delta)(1-\nabla) = 1$
- $\delta = E^{\frac{1}{2}} - E^{\frac{-1}{2}}$
- $\mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{\frac{-1}{2}}\right)$

Advancing difference formula:

If y= f(x) is a function in x then $f(a+nh)A = n_{c_0}f(a) + n_{c_1}\Delta f(a) + n_{c_2}\Delta^2 f(a) + \cdots + n_{c_n}\Delta^n f(a)$.

Fundamental theorem of difference calculus:

If f(x) is a polynomial of n^{th} degree in x, then the n^{th} difference of f(x) is constant and $(n+1)^{th}$ difference is zero.

Factorial function:

The factorial function $x^{(n)}$ is defined as $x^{(n)} = x (x – h) (x – 2h) \ldots (x – (n-1)h)$ for n = 1,2,3, … and $x^{(0)}$ is defined as 1.

If the interval of differencing is 1, then $x^{(n)} = x (x – 1) (x – 2) \ldots (x – (n-1))$.

Interpolation with Equal Intervals

Newton – Gregory formula for forward interpolation with equal intervals:

If y = f(x) is a function which assumes the values y_0 , y_1 , y_2 ... y_n for (n+1) equidistant values x_0 , x_1 , x_2 , ... x_n where x_0 = a , x_1 = a + h , x_2 = a + 2h , ... x_n = a + nh of the argument x, then $y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)(u-2)\ldots(u-(n-1))}{n!}\Delta^n y_0$ where $\mathbf{u} = \frac{x-x_0}{h}$ **OR**

$$f(a+nh) = f(a) + \frac{u^{(1)}}{1!}\Delta f(a) + \frac{u^{(2)}}{2!}\Delta^2 f(a) + \frac{u^{(3)}}{3!}\Delta^3 f(a) + \cdots \frac{u^{(n)}}{n!}\Delta^n f(a).$$

Newton – Gregory formula for backward interpolation with equal intervals:

If y = f(x) is a function which assumes the values y_0 , y_1 , y_2 ... y_n for (n+ 1) equidistant values x_0 , x_1 , x_2 , ... x_n where x_0 = a , x_1 = a + h , x_2 = a + 2h , ... x_n = a + nh of the argument x, then $y_u = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots + \frac{u(u+1)(u+2)\ldots(u+(n-1))}{n!}\nabla^n y_n$

where $\mathbf{u} = \frac{x-x_n}{h}$

Gauss's forward formula for equal intervals:

If y = f(x) is a function , which takes the values . . . y_{-2} , y_{-1} , y_0 , y_1 , y_2 , . . . corresponding to the values of x = . . . $x_0 – 2h$, $x_0 – h$, x_0 , $x_0 + h$, $x_0 + 2h$, . . . then

$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots$ Where $u = \frac{x-x_0}{h}$

Gauss's backward formula for equal intervals:

If y = f(x) is a function, which takes the values . . . y_{-2} , y_{-1} , y_0 , y_1 , y_2 , . . . corresponding to the values of x = . . . $x_0 - 2h$, $x_0 - h$, x_0 , $x_0 + h$, $x_0 + 2h$, . . . then

$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + . . .$ Where u = $\frac{x-x_0}{h}$.

Stirling's difference formula:

If y = f(x) is a function, which takes the values . . . y_{-2} , y_{-1} , y_0 , y_1 , y_2 , . . . corresponding to the values of x = . . . $x_0 - 2h$, $x_0 - h$, x_0 , $x_0 + h$, $x_0 + 2h$, . . . then

$y_u = y_0 + u\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + . . .$ Wher u = $\frac{x-x_0}{h}$.

Bessel's difference formula:

If y = f(x) is a function, which takes the values . . . y_{-2} , y_{-1} , y_0 , y_1 , y_2 , . . . corresponding to the values of x = . . . $x_0 - 2h$, $x_0 - h$, x_0 , $x_0 + h$, $x_0 + 2h$, . . . then

$y_u = \frac{y_0+y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left(\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right) + \frac{u\left(u-\frac{1}{2}\right)(u-1)}{3!}\Delta^3 y_{-1} +$
$\frac{u(u^2-1)(u-2)}{4!}\left(\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}\right) + \frac{u\left(u-\frac{1}{2}\right)(u^2-1)(u-2)}{5!}\Delta^5 y_{-2} + . . .$ Where u = $\frac{x-x_0}{h}$.

Everett's difference formula:

If y = f(x) is a function, which takes the values . . . y_{-2} , y_{-1} , y_0 , y_1 , y_2 , . . . corresponding to the values of x = . . . $x_0 - 2h$, $x_0 - h$, x_0 , $x_0 + h$, $x_0 + 2h$, . . . then

$y_u = (1-u)y_0 + uy_1 - \frac{u(u-1)(u-2)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 - \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^4 y_{-2} +$
$\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \cdots$ Where u = $\frac{x-x_0}{h}$.

Interpolation with Unequal Intervals

Divided Differences:

Let y = f(x) be a function which assumes the values $f(x_0)$, $f(x_1)$, $f(x_2)$, … $f(x_n)$ corresponding to the values of $x_0, x_1, x_2, \ldots x_n$ of the argument x, where the intervals $x_0 - x_1$, $x_1 - x_2$, …, $x_{n-1} - x_n$ are not necessarily equal.

The first divided difference of f(x) for the arguments x_{n-1}, x_n is defined as

$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{(x_n - x_{n-1})}.$$

The second divided difference of f(x) for the arguments x_{n-2}, x_{n-1}, x_n is defined as

$$f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{(x_n - x_{n-2})}$$

In general the n^{th} divided difference for the arguments $x_0, x_1, x_2, \dots x_n$ is defined as

$$f(x_0, x_1, \dots, x_n) = \frac{f(x_1, x_2, \dots x_n) - f(x_0, x_1, \dots x_{n-1})}{(x_n - x_0)}$$

Newton's Divided difference formula:

Let $y = f(x)$ be a function which assumes the values $f(x_0)$, $f(x_1)$, $f(x_2)$, … $f(x_n)$ corresponding to the values of $x_0, x_1, x_2, \dots x_n$ of the argument x which are not equally spaced, then

$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) + \dots + (x - x_0)(x - x_1)\dots(x - x_{n-1})\, f(x_0, x_1, x_2, \dots x_n)$.

Lagrange's Interpolation formula:

Let $y = f(x)$ be a function which assumes the values $f(x_0)$, $f(x_1)$, $f(x_2)$, … $f(x_n)$ corresponding to the values of $x_0, x_1, x_2, \dots x_n$ of the argument x which are not equally spaced, then

$$f(x) = \frac{(x - x_1)(x - x_2)\dots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\dots(x_0 - x_n)} f(x_0) + \frac{(x - x_0)(x - x_2)\dots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\dots(x_1 - x_n)} f(x_1) + \cdots + \frac{(x - x_0)(x - x_1)\dots(x - x_{n-1})}{(x_n - x_1)(x_n - x_2)\dots(x_n - x_{n-1})} f(x_n)$$

Numerical Differentiation

Def: Let $y = f(x)$ be the given function. The process of evaluating the derivatives of a function $f(x)$ with the help of the given set of values of that function is known as numerical differentiation.

Derivative using Newton's forward interpolation formula:

- $\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \cdots\right]$
- $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \cdots\right]$
- $\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$
- $\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right]$

Derivative using Newton's backward interpolation formula:

- $\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + \cdots\right]$
- $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \cdots\right]$
- $\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$
- $\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n - \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$

Derivative using Stirling's formula:

- $\frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} + u\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{4u^3-2u}{4!}\Delta^4 y_{-2} + \cdots\right]$
- $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{6u^2-1}{12}\Delta^4 y_{-2} + \cdots\right]$
- $\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \cdots\right]$
- $\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\right]$

Derivative using Newton's divided difference formula:

If the values of x are not equally spaced, we use Newton's divided difference interpolation formula. Newton's divided interpolation formula is

$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) + \ldots + (x - x_0)(x - x_1)\ldots(x - x_{n-1})\, f(x_0, x_1, x_2, \ldots x_n)$.

Differentiate this equation w.r.t. x as many times as we require and put $x = x_i$, we get the required derivatives.

Numerical Integration

Numerical Quadrature:

Numerical integration is the process of evaluating a definite integral $\int_a^b f(x)dx$ from a set of known numerical values of the integrand f(x). If it is applied to the integration of a function of single variable, the process is known as quadrature.

General Quadrature formula for equidistant ordinates (OR) Newton Cote's Quadrature formula:

Let y = f(x) be a function which takes the values y_0 , y_1 , y_2 , . . . y_n corresponding to the values of the argument $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... $x_n = a + nh = b$ so that $b - a = nh$. Then

$$\int_a^b y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!}\right.$$

$$\left. + \cdots upto\ (n+1) terms\right]$$

The Trapezoidal Rule:

Let y = f(x) be a function which takes the values y_0 , y_1 , y_2 , . . .y_n corresponding to the values of the argument $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... $x_n = a + nh = b$ so that b – a = nh. Then

$$\int_a^b f(x)dx = \frac{h}{2}[\,(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\,]$$

Simpson's One-Third rule:

Let y = f(x) be a function which takes the values y_0 , y_1 , y_2 , . . .y_n corresponding to the values of the argument $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... $x_n = a + nh = b$ so that b – a = nh. Then

$$\int_a^b f(x)dx = \frac{h}{3}[\,(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})\,]$$

Simpson's Three Eighth rule:

Let y = f(x) be a function which takes the values y_0 , y_1 , y_2 , . . .y_n corresponding to the values of the argument $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... $x_n = a + nh = b$ so that b – a = nh. Then

$$\int_a^b f(x)dx = \frac{3h}{8}[\,(y_0 + y_n) + 3(y_1 + y_2 + y_4 + \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})\,]$$

Boole's rule:

Let y = f(x) be a function which takes the values y_0 , y_1 , y_2 , . . .y_n corresponding to the values of the argument $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... $x_n = a + nh = b$ so that b – a = nh. Then

$$\int_a^b f(x)dx = \frac{2h}{45}[\,7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + \cdots + 7y_n\,]$$

Weddle's Rule:

Let y = f(x) be a function which takes the values y_0 , y_1 , y_2 , . . .y_n corresponding to the values of the argument $x_0 = a$, $x_1 = a + h$, $x_2 = a + 2h$, ... $x_n = a + nh = b$ so that b – a = nh. Then

$$\int_a^b f(x)dx = \frac{3h}{10}[\,y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + \cdots + y_n\,]$$

Numerical Solution of Ordinary Differential Equations

Taylor Series:

We consider the differential equation y' = f(x,y) with initial condition $y(x_0) = y_0$.

If y(x) is the exact solution of given equation, then the Taylor's series for y(x) around $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \cdots$$

Picard's method of successive approximation:

We consider the differential equation y' = f(x,y) with initial condition $y(x_0) = y_0$.

If y(x) is the exact solution of given equation, then the Picard's method is

$$y^{(n)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(n-1)}\right)dx$$

Euler's method:

Consider the first order differential equation y' = f(x,y) with $y = y_0$ when $x = x_0$.

We wish to solve the given equation, for values of y at $x = x_i$ where $x_i = x_0 + ih$, i = 1, 2, 3, …. then

$$y_{n+1} = y_n + h\,f(x_n, y_n).$$

Modified Euler's Method:

Consider the first order differential equation y' = f(x,y) with $y = y_0$ when $x = x_0$.

We wish to solve the given equation, for values of y at $x = x_i$ where $x_i = x_0 + ih$, i = 1, 2, 3, …. then

By Euler's method $y_1^{(0)} = y_0 + h\,f(x_0, y_0)$.

By Modified Euler's method $y_1^{(n+1)} = y_0 + \frac{h}{2}\left[\,f(x_0, y_0) + f\left(x_1, y_1^{(n)}\right)\right]$ for n = 0,1,2,… where $y_1^{(n)}$ is the n^{th} approximation to y_1.

Runge – Kutta method:

Consider the first order differential equation y' = f(x,y) with $y = y_0$ when $x = x_0$.

We wish to solve the given equation, for values of y at $x = x_i$ where $x_i = x_0 + ih$, i = 1, 2, 3, ….

then

First – order Runge – Kutta method

$$y_1 = y_0 + h\, f(x_0, y_0)$$

Second – order Runge – Kutta method

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) \text{ where } k_1 = h\, f(x_0, y_0) \ and\ k_2 = h\, f(x_0 + h, y_0 + k_1)$$

Third – order Runge – Kutta method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Where $k_1 = h\, f(x_0, y_0)$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f(x_0 + h, y_0 + 2k_2 - k_1)$$

Fourth -Order Runge – Kutta method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Where $k_1 = h\, f(x_0, y_0)$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

Milne's Predictor formula:

Consider the first order differential equation y' = f(x,y) with $y = y_0$ when $x = x_0$.

Milne's predictor formula is

$$y_{n+1}^{p} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) = y_{n-3} + \frac{4h}{3}(2f_{n-2} - f_{n-1} + 2f_n)$$

Milne's Corrector formula:

Consider the first order differential equation y' = f(x,y) with $y = y_0$ when $x = x_0$.

Milne's Corrector formula is

$$y_{n+1}^{c} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1}) = y_{n-1} + \frac{h}{3}\left(f_{n-1} + 4f_n + f_{n+1}^{p}\right)$$

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